Concave Generalized Flows with Applications to Market Equilibria

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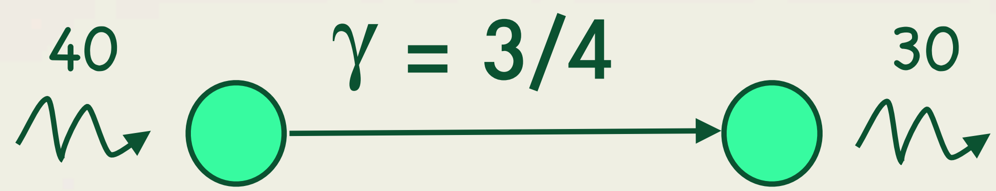
Generalized Flows

* Network flow model, with gain factors on the arcs.



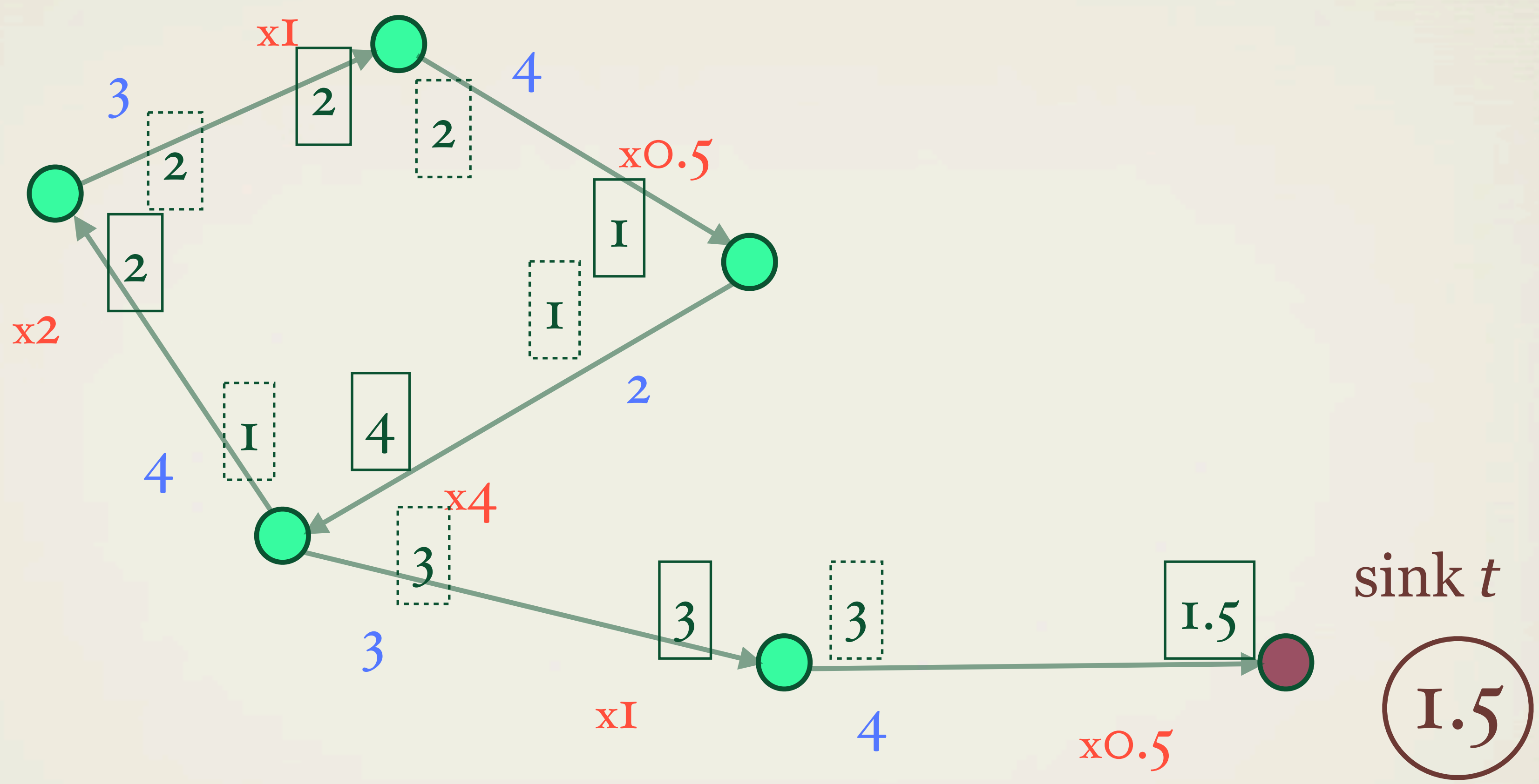
* Maximize the flow amount reaching the sink t.
* Introduced by Kantorovich '39, Dantzig '62.
* Several applications: financial analysis, transportation, management, etc.

Generalized Flows

Capacities

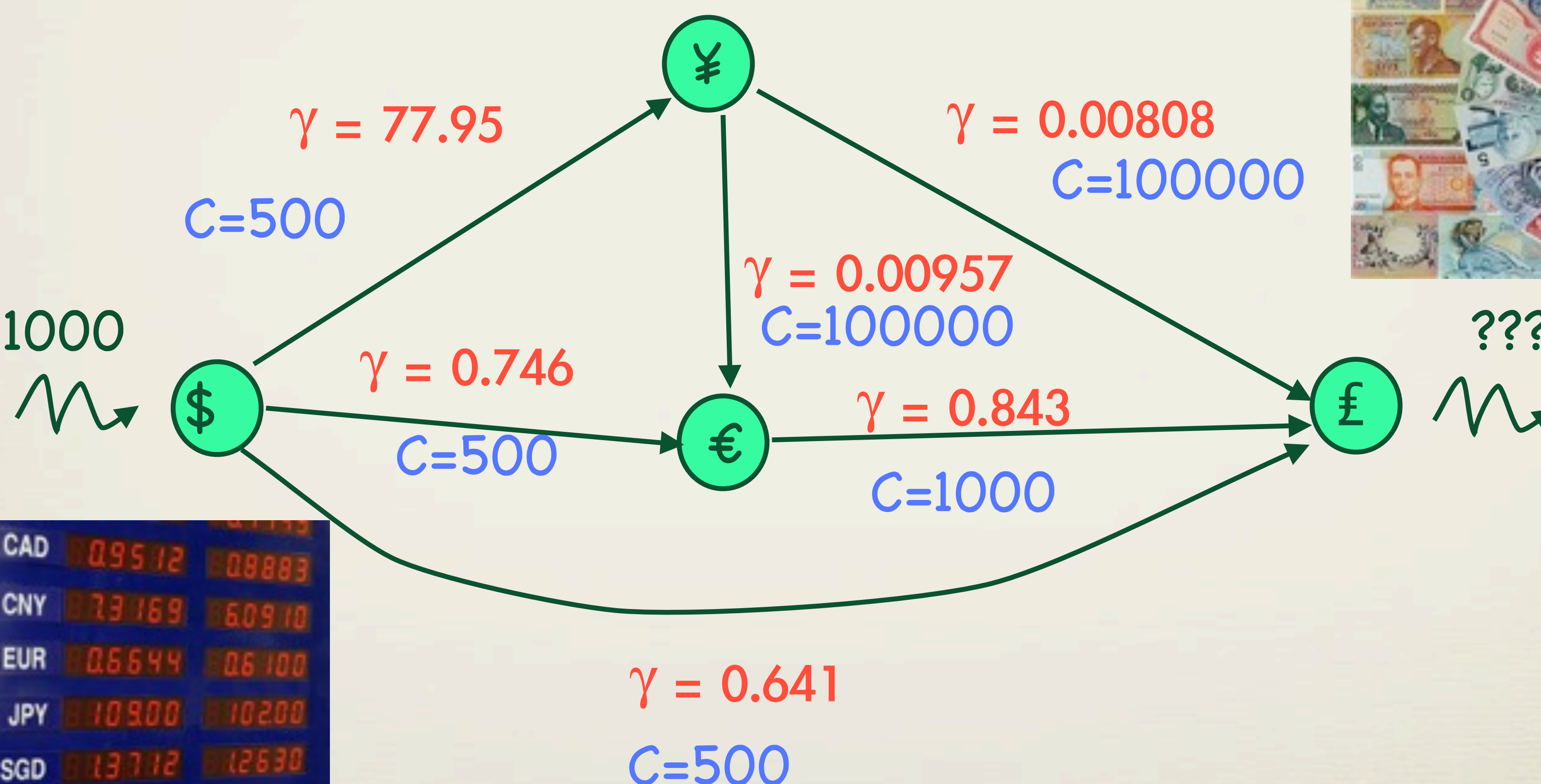
Gain factors

Flow



Generalized Flows

* Currency conversion with bounds: obtain the most £ from 1000$.



Generalized Flows

* Linear program.
* Early combinatorial algorithms: Onaga '66, Truemper '77.
* First polynomial time combinatorial algorithm: Goldberg, Plotkin, Tardos '91.
* Followed by Cohen, Megiddo '94, Goldfarb, Jin '96, Goldfarb, Jin, Orlin '97, Tardos, Wayne '98, Wayne '02, Radzik '04, Restrepo, Williamson '09, etc.

Concave Generalized Flows

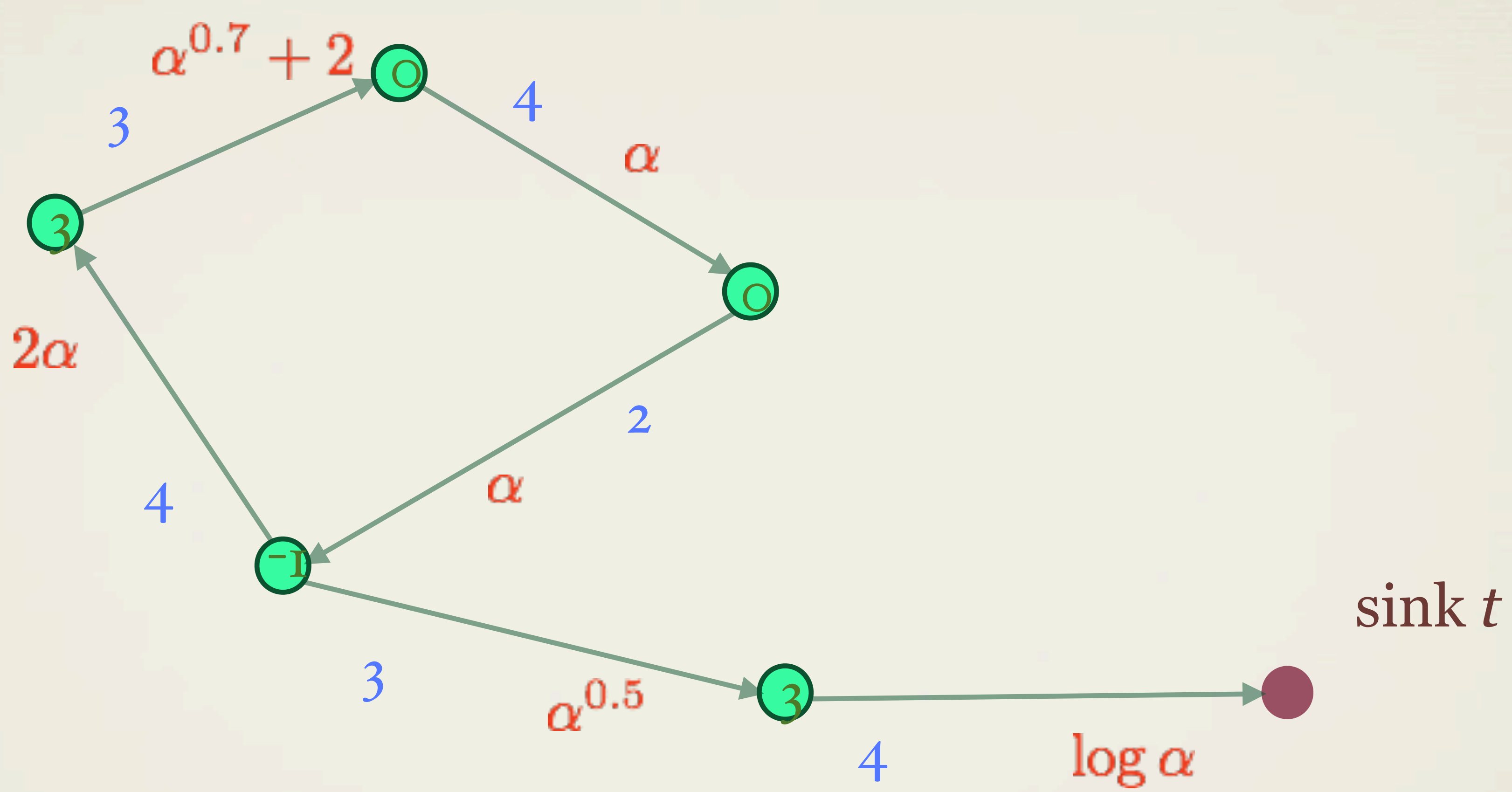
* Instead of gain factors, concave increasing gain functions.

Generalized Flows

Capacities

Gain functions

Node demands



Monday, April 9, 2012

Convex Program

$$\max \sum_{j:jt \in E} \Gamma_{jt}(f_{jt}) - \sum_{j:tj \in E} f_{tj}$$

$$\sum_{j:ji \in E} \Gamma_{ji}(f_{ji}) - \sum_{j:ij \in E} f_{ij} \geq b_i \quad \forall i \in V - t$$

$$\ell_{ij} \leq f_{ij} \leq u_{ij} \quad \forall ij \in E$$

Concave Generalized Flows

- First defined by Truemper '78.
- Solvable via general purpose convex solver.
- Shigeno '06 gave a combinatorial algorithm that is polynomial for some special classes of gain functions, including piecewise linear.
- We present a polynomial combinatorial algorithm for finding an ε-approximate solution in running time

$$O(m(m + \log n) \log(MUm/\varepsilon))$$

- For problems with a rational optimal solution, we can find it in polynomial time with a final rounding.

Linear Fisher markets

- B: buyers, G: goods.
- Buyer i has budget m_i, 1 divisible unit of each good j.
- Utility U_{ij} for buyer i on 1 unit of good j.
- Market clearing: prices p_j and allocations x_{ij} if:
  - everything is sold
  - all money is spent
  - only best bang-per-buck purchases: max. U_{ij}/p_j.

Eisenberg-Gale convex program, 1959

$$\max \sum_{i \in B} m_i \log U_i$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \qquad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \qquad \forall j \in G$$

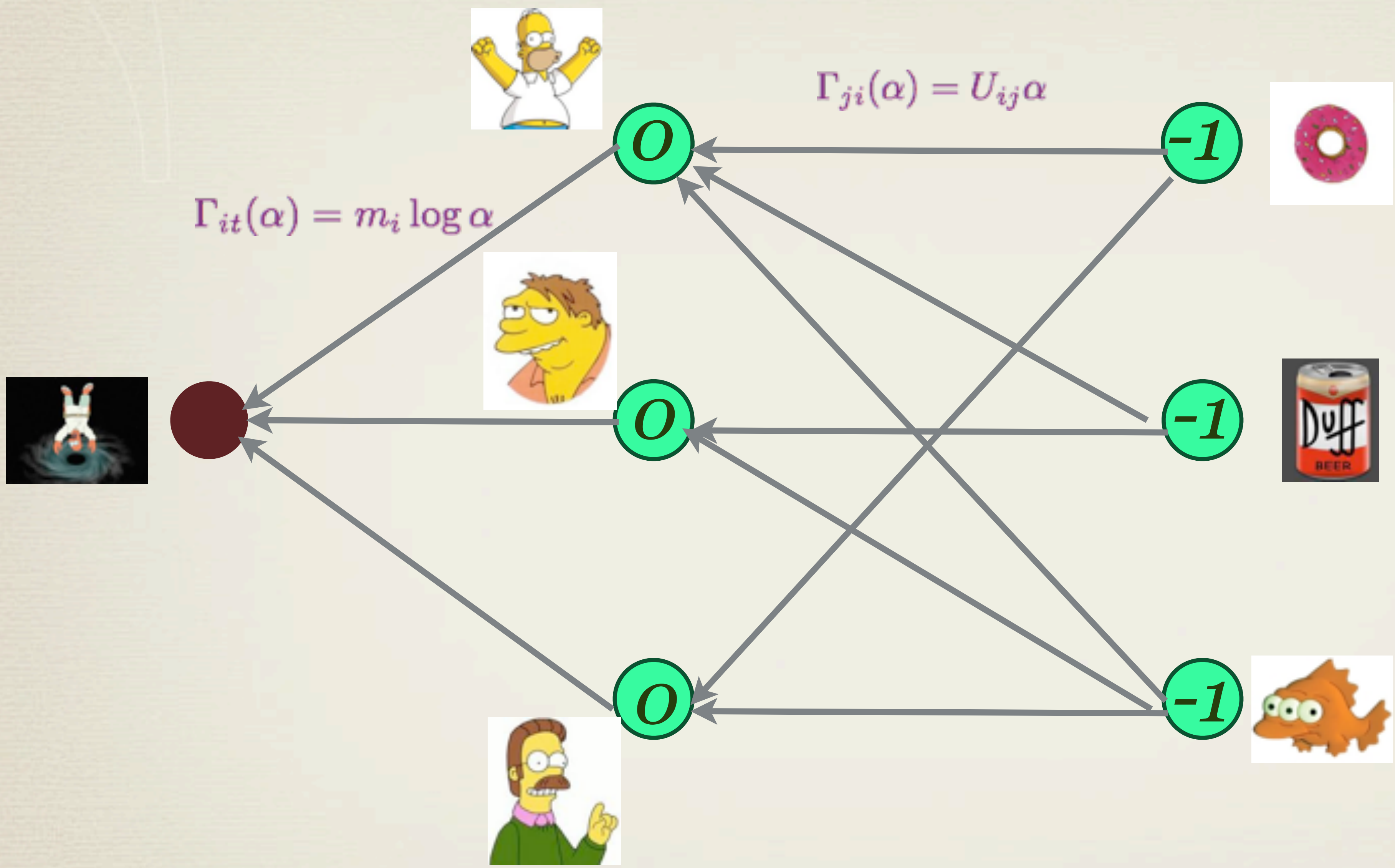
$$x_{ij} \geq 0 \qquad \forall i \in B, j \in G$$

* Optimal solution corresponds to equilibrium prices.
* There exists a rational optimal solution

Combinatorial algorithms for linear Fisher markets

- Devanur, Papadimitriou, Saberi, Vazirani ’02: polynomial time combinatorial algorithm.
- Strongly polynomial algorithms: Orlin ’10, V. ’12.
- Several extensions and generalizations studied.

Reduction to Concave Generalized Flows



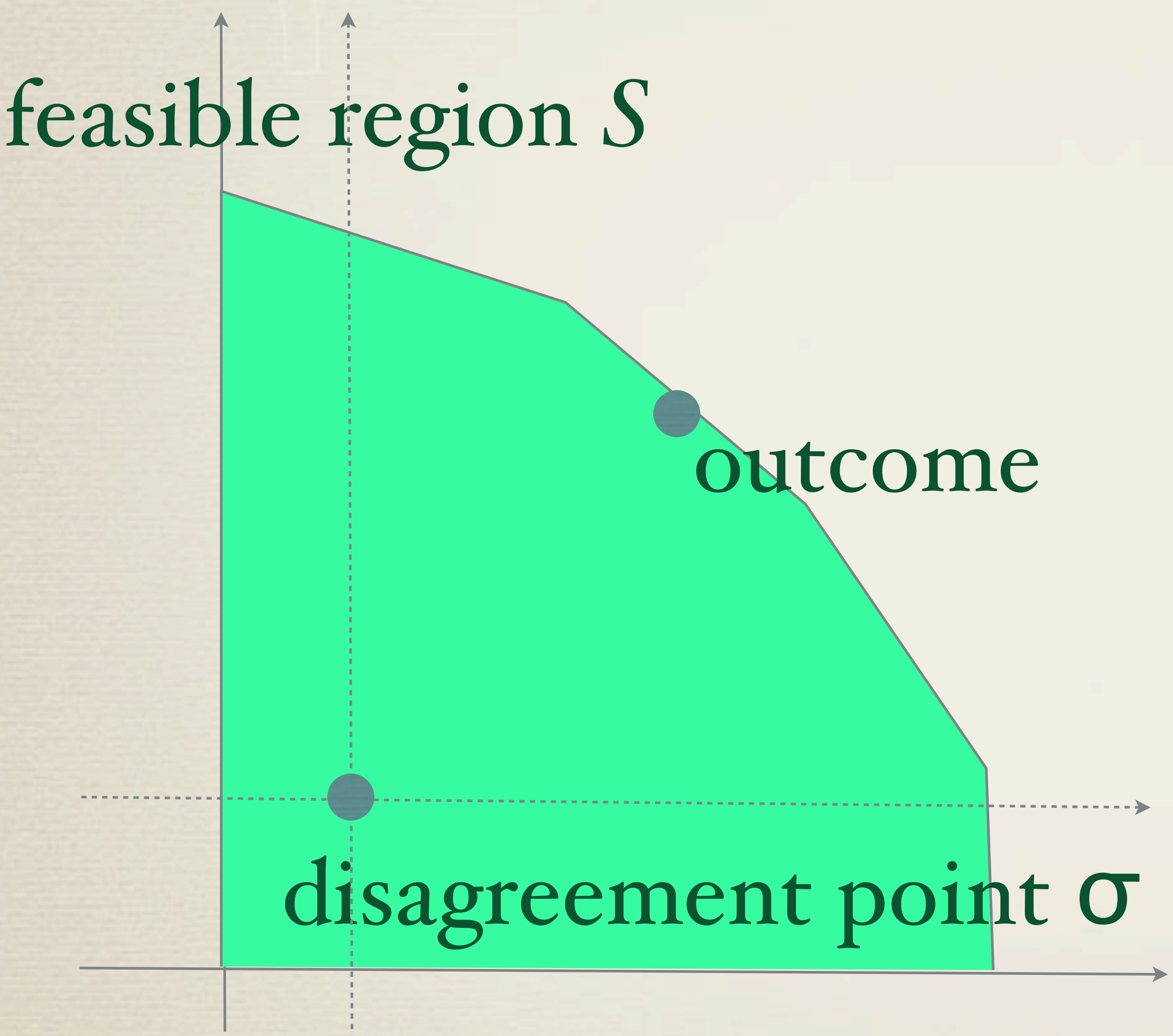
Extensions of linear Fisher markets

- Goel, Vazirani '10: perfect price discrimination
  - Replace $\Gamma_{ji}(\alpha) = U_{ij}\alpha$ by a piecewise linear concave function.
  - Using our model, it can be replaced by arbitrary concave!

Nash bargaining, 1950

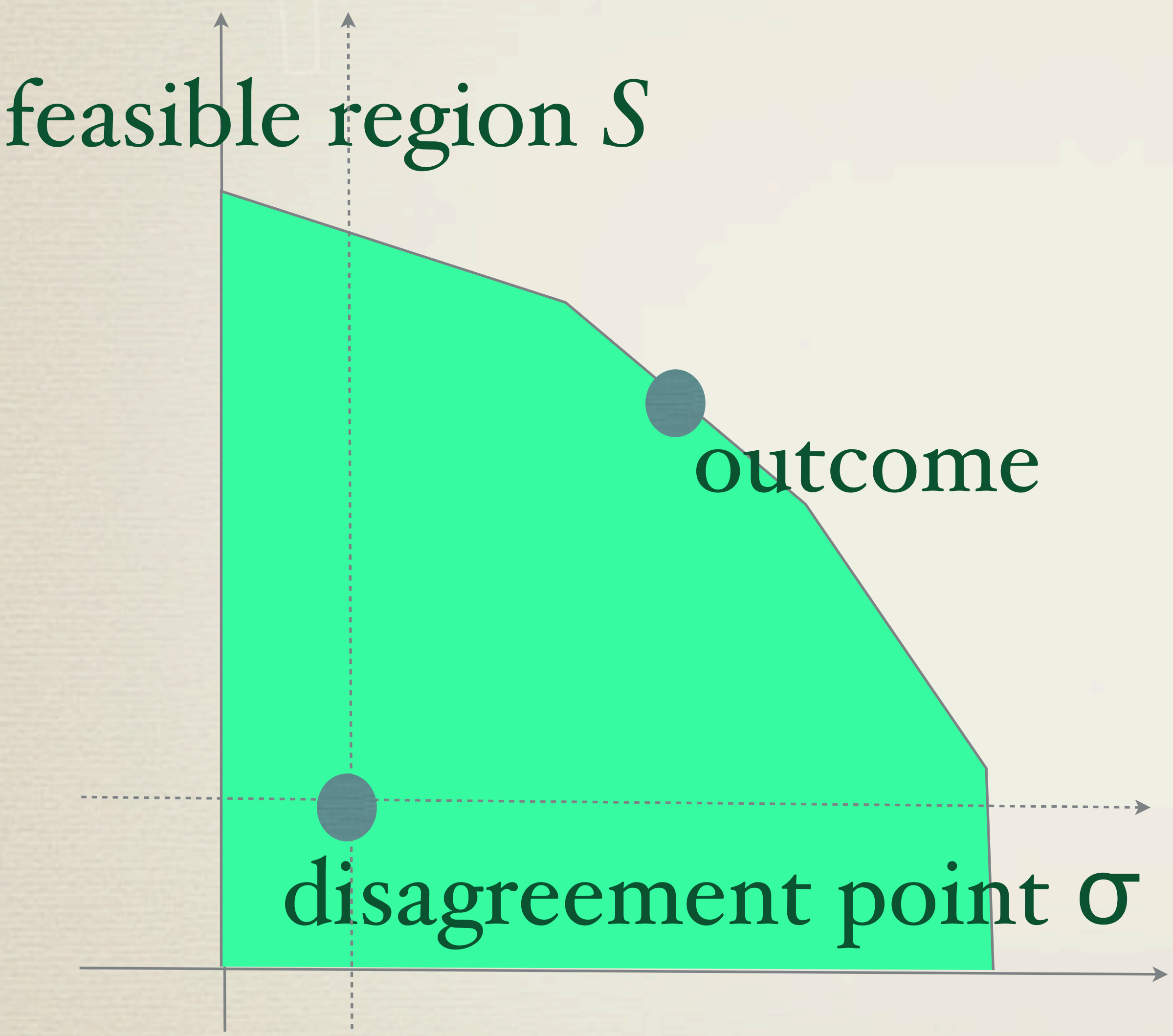
* n players, set of possible outcomes $S \subseteq \mathbb{R}^n_+$
* In outcome $s = (s_1, \ldots, s_n) \in S$, player i gets utility s_i.
* Disagreement point (status quo): $\sigma \in S$
* The players have to agree together in an outcome. If they cannot agree, the status quo remains.

Nash bargaining, 1950



- Which is the best outcome?
- Four criteria:
  - Pareto optimality
  - Invariance under affine transformations
  - Symmetry
  - Indifference of independent alternatives

Nash bargaining, 1950



Theorem (Nash, 1950)

For a convex feasible region, there exists a unique optimal solution, the one maximizing

$$\sum_{i\in[n]} \log(s_i - \sigma_i)$$

Arrow-Debreu Nash bargaining: Vazirani '12

- Nash bargaining between agents, each of them having an initial endowment of goods, giving utility c_i to player i.
- Possible outcomes: distributions of goods.

$$\max \sum_{i \in B} \log(U_i - c_i)$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$

Arrow-Debreu Nash bargaining: Vazirani '12

* Vazirani '12: sophisticated two phase algorithm, first deciding feasibility, then optimality.

$$\max \sum_{i \in B} \log(U_i - c_i)$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

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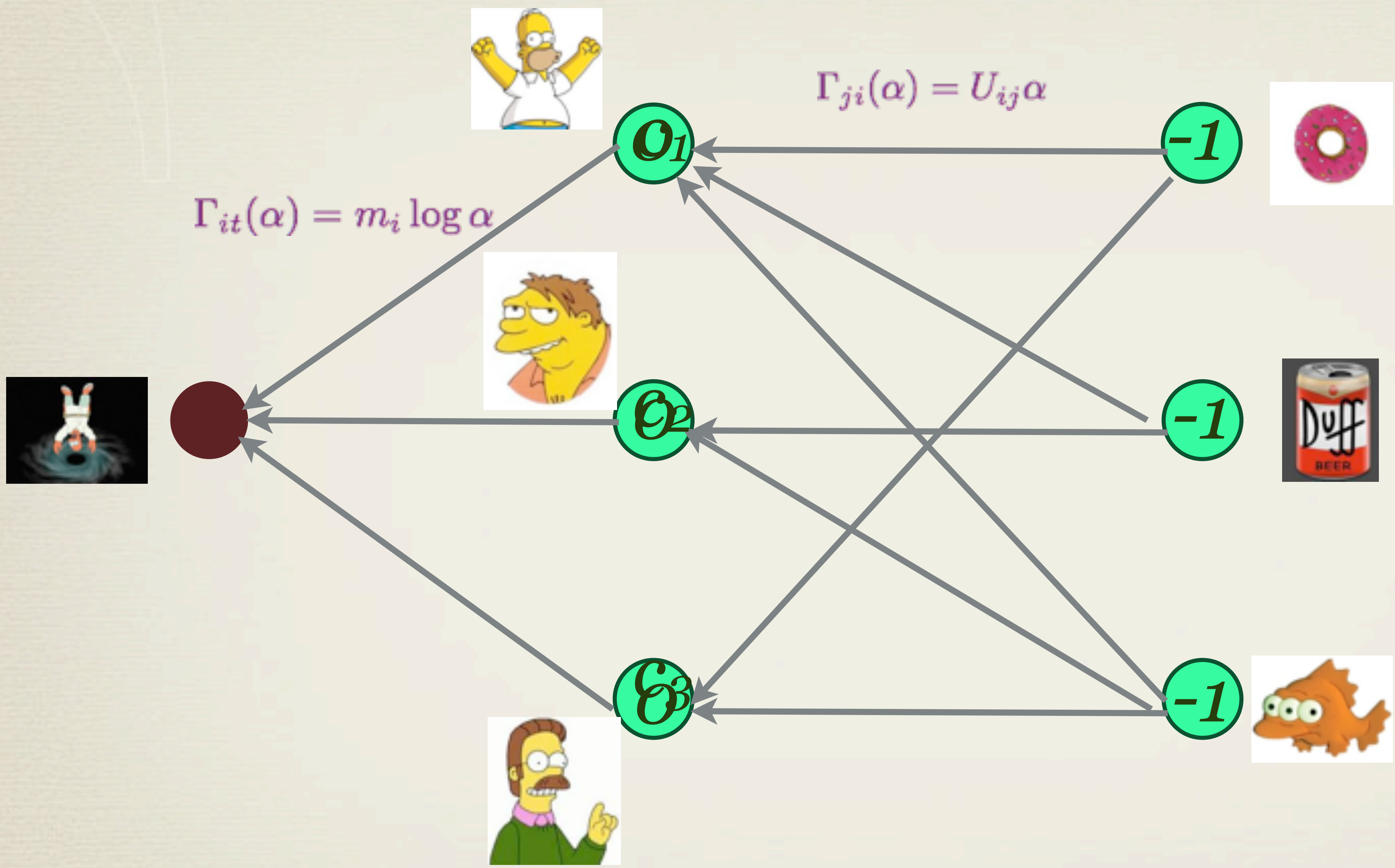
Reduction to Concave Generalized Flows



Arrow-Debreu Nash bargaining: Vazirani, '12

$$\max \sum_{i \in B} m_i \log(U_i - c_i)$$

$$U_i \leq \sum_{j \in G} U_{ij} x_{ij} \quad \forall i \in B$$

$$\sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G$$

$$x_{ij} \geq 0 \quad \forall i \in B, j \in G$$

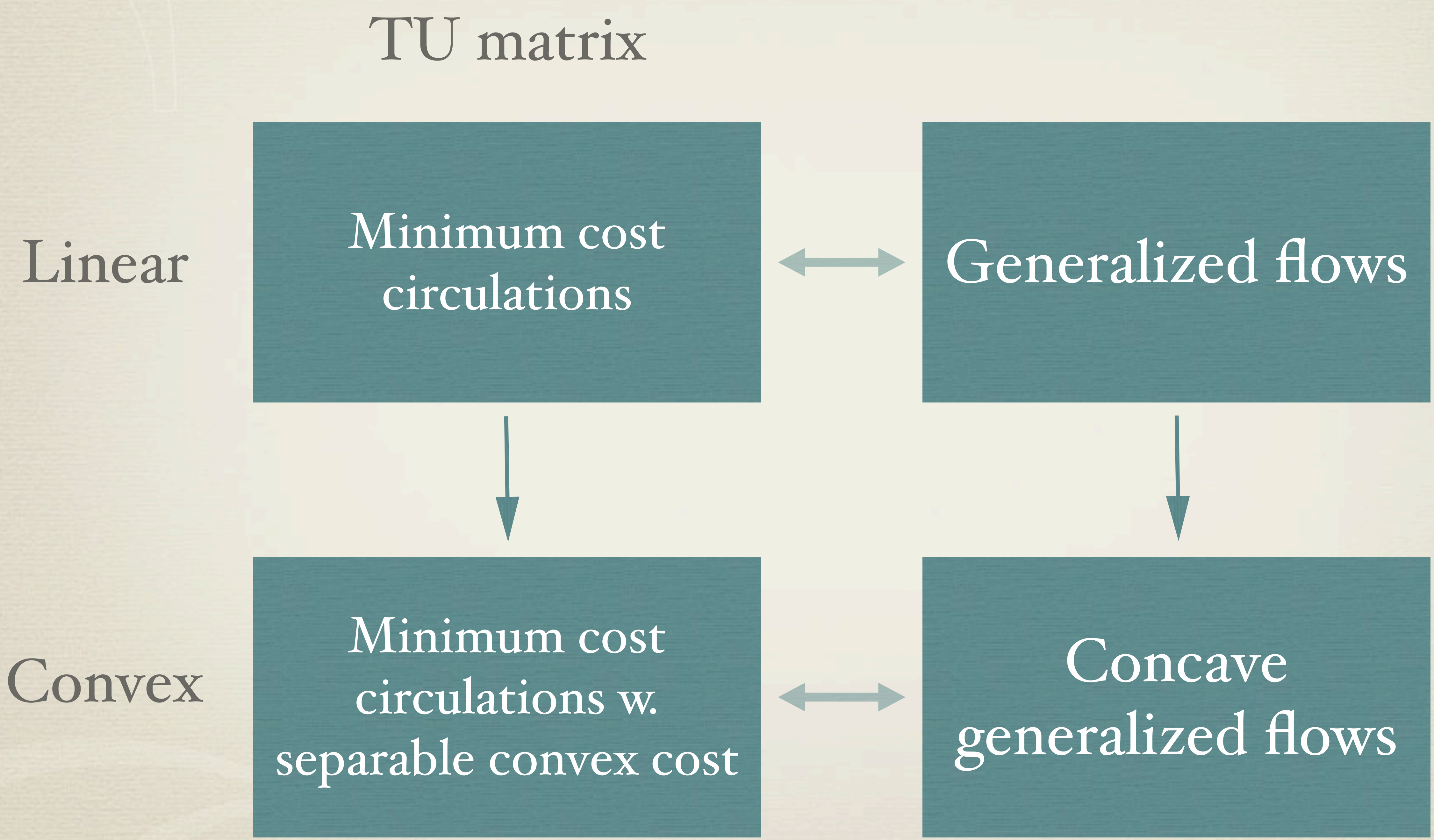
* Nonsymmetric Nash bargaining: Kalai '77
* Different weights m_i for player i.
* Finding a combinatorial algorithm was left open. Our model also captures this, solving in

$$O(m^2(\log C_{\max} + n \log(n U_{\max} M_{\max})))$$

* Vazirani '12 for symmetric:

$$O(n^8 \log U_{\max} + n^4 \log C_{\max})$$

Linear and convex flow problems



Linear and convex flow problems

	TU matrix	
Linear	Minimum cost circulations	Generalized flows
Convex	Minimum cost circulations w. separable convex cost	Concave generalized flows

Minimum cost circulations: Two main algorithmic paradigms

Cycle canceling

- always maintain a feasible circulation
- cancel negative cycles in the residual graph

Successive shortest paths

- always maintain dual optimality: the residual graph contains no negative cycles.
- primal feasibility is violated: nodes may have excess/demand.
- send flow from a node w/ excess to a node w/ demand on a shortest path in the residual graph.

Minimum cost circulations: Two main algorithmic paradigms

* The basic algorithm is not polynomial (possibly not even finite!) in either framework.
* Majority of efficient algorithms are based on either paradigm.
* For successive shortest paths, first polynomial version: Edmonds, Karp '72.

Minimum cost flows:
Edmonds, Karp '72

* **Scaling algorithm:** first transport the large flows, then go for the rest.
* **Scaling parameter Δ:** flow sent in Δ units, decreasing by a factor 2 between two phases.
* **Δ-residual graph $G_{f,\Delta}$:** arcs w/ residual capacity at least Δ.
* Invariant: $G_{f,\Delta}$ does not contain any negative cycles.
* Send Δ units of flow from positive nodes with excess $\geq\Delta$ to negative nodes with deficiency $\geq\Delta$ on a shortest path in $G_{f,\Delta}$.
* At the beginning of the $\Delta/2$ phase, the invariant can be violated because $G_{f,\Delta/2}$ can contain new arcs. Saturate these arcs by creating at most $m\Delta/2$ new excess.

Linear and convex flow problems

	TU matrix	
Linear	Minimum cost circulations	Generalized flows
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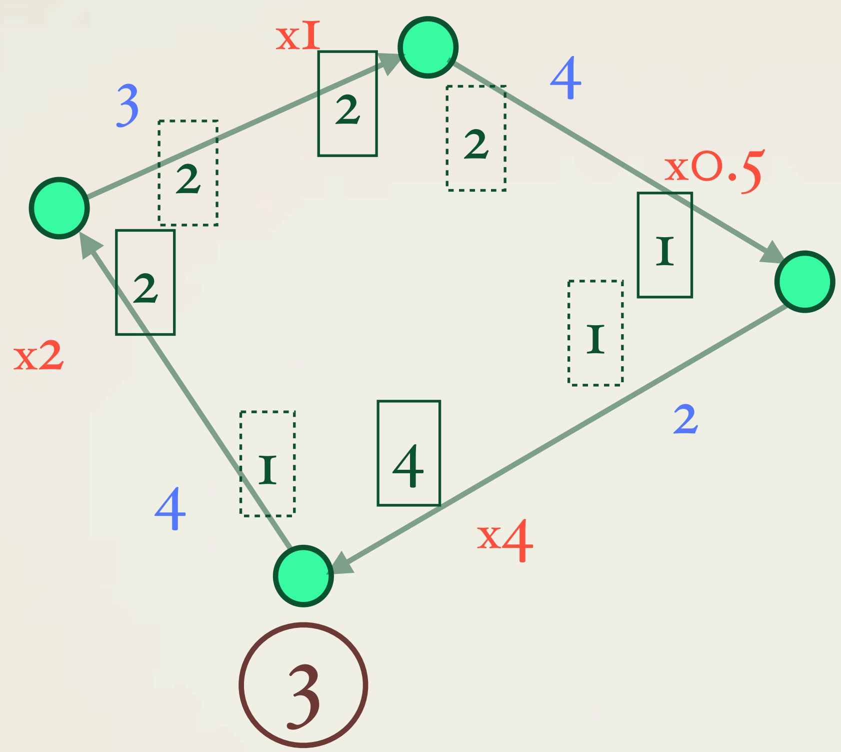
Flow generating cycle

Capacities

Gain factors

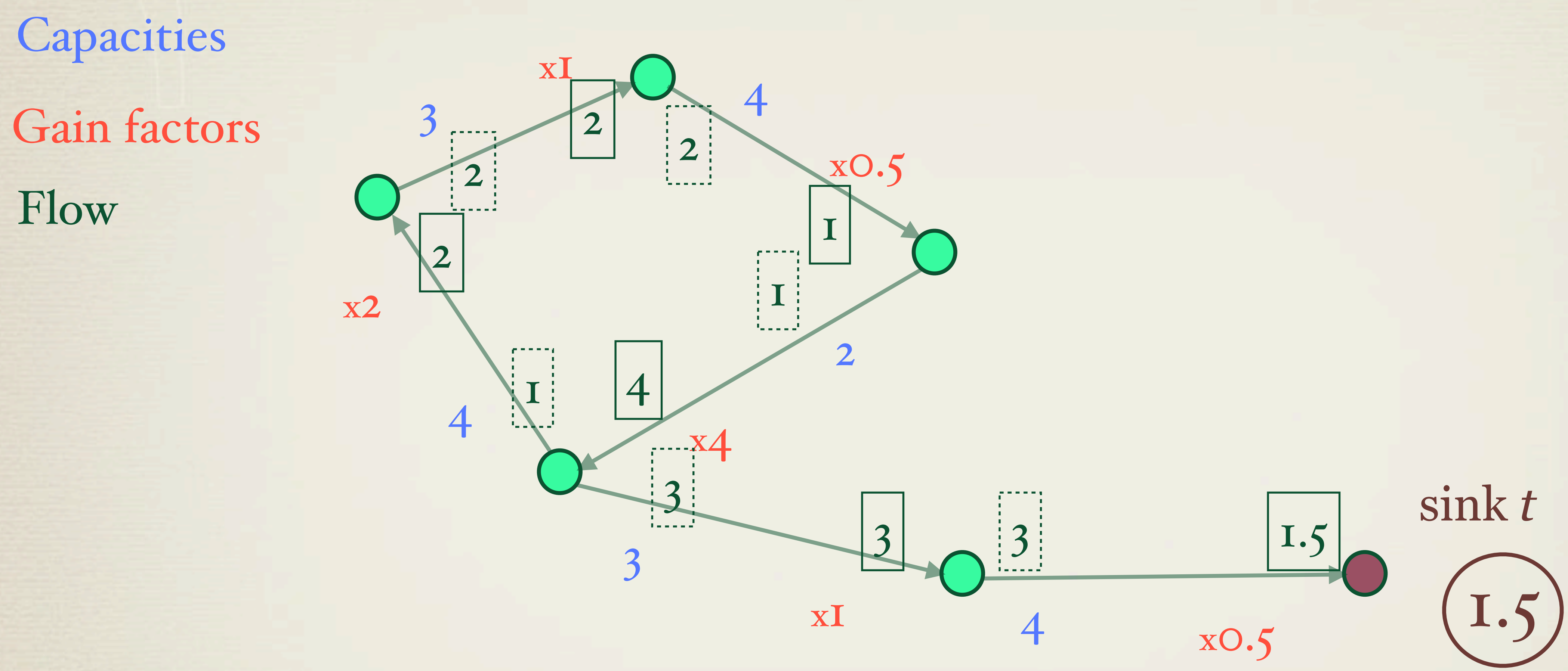
Flow



Cycle C in E_f is **flow generating**, if

$$\gamma(C) = \Pi_{ij \in C} \gamma_{ij} > 1$$

Generalized augmenting path (GAP) = Flow generating cycle + dir. path to the sink



Claim:

The flow is optimal ⇔ there exists no GAP in the residual graph.

Connection to minimum cost circulations

Observation: C is a flow generating cycle $\Leftrightarrow$ C is a negative cycle for the cost function.

$$c_{ij} = -\log \gamma_{ij}.$$

Corollary: All flow generating cycles can be canceled by a minimum cost circulation algorithm.

Question: How can we send the generated flow to the sink?

Connection to minimum cost circulations

Cycle canceling aspect:

Generating flow around cycles.

negative cycle ↔ flow generating cycle

Successive shortest paths aspect:

Send the generated flow to the sink.

shortest path ↔ highest gain augmenting path

Most polynomial algorithms use a combination of the two paradigms.

Linear and convex flow problems

	TU matrix	
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Minimum convex cost circulations

* Both the cycle canceling and successive shortest path paradigms provide polynomial time algorithms

Cycle canceling:

Karzanov&McCormick '97

Successive shortest paths:

Minoux 86, Hochbaum&Shantikumar '94

Minimum convex cost circulations

* The convex functions are represented by oracles. The two approaches use different relaxations based on different type oracles.
    * value oracle: given α, return $C_{ij}(\alpha)$ *(Minoux)*.
    * derivative oracle: given α, return $C'_{ij}(\alpha)$. *(Karzanov&McCormick)*

Linear and convex flow problems

	TU matrix	
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First attempt: Shigeno ’06
Extend Fat Paths algorithm by Goldberg et al.

Idea:

1. Cycle canceling via the Karzanov-McCormick method.
2. Carry the excess flow to the sink via the Minoux method.

Problem:

1. The two paradigms need different type of approximation of the concave function.
2. Cycle canceling can only be done approximately.

First attempt: Shigeno '06

Extend Fat Paths algorithm by Goldberg et al.

Main obstacle: in general, given Δ, there exists no general bound on ε so that

$$\pi(j) - \pi(i) - \varepsilon \le C'_{ij}(f_{ij})$$

guarantees

$$\pi(j) - \pi(i) \le \frac{C_{ij}(f_{ij}+\Delta)-C_{ij}(f_{ij})}{\Delta}$$

First attempt: Shigeno '06

Extend Fat Paths algorithm by Goldberg et al.

- To avoid conflict of relaxations, we want to use only one of the two paradigms.
- Problem:
  - pure cycle canceling did not seem adaptable.
  - there was no pure successive shortest path method.
  - closest: Goldfarb, Jin, Orlin '97: single cycle canceling phases, only path augmentations for the rest.

Solution

* To develop a purely scaling algorithm for linear generalized flows, we

  redefine the problem.

* To extend it to concave gain functions.

  redefine the number zero.

Symmetric formulation

* No designated sink node.
* Violation of conservation penalized at possibly different rates.
* Initial transformation: lower bounds set to 0, upper bounds removed.

$$\min \sum_{i \in V} M_i \kappa_i$$

$$\sum_{j:ji \in E} \gamma_{ji} f_{ji} - \sum_{j:ij \in E} f_{ij} \geq b_i - \kappa_i \quad \forall i \in V$$

$$0 \leq f_{ij} \quad \forall ij \in E$$

$$0 \leq \kappa_i \quad \forall i \in V$$

Symmetric formulation

Modelling the sink formulation: set M_t=*1*, and M_i very large for $i \neq t$. Also set b_t very large *("very"=polynomially)*.

$$\min \sum_{i \in V} M_i \kappa_i$$

$$\sum_{j:ji \in E} \gamma_{ji} f_{ji} - \sum_{j:ij \in E} f_{ij} \geq b_i - \kappa_i \quad \forall i \in V$$

$$0 \leq f_{ij} \quad \forall ij \in E$$

$$0 \leq \kappa_i \quad \forall i \in V$$

Symmetric formulation

- The highest gain augmenting path algorithm by Goldfarb, Jin & Orlin first cancels all flow generating cycles.
- Afterwards, they only augment on highest gain augmenting paths: no new flow generating cycles is created.
- Polynomial running time: scaling + clever bookkeeping *(arc imbalances)*.

Symmetric formulation

* Using the symmetric formulation instead, we can remove the initial cycle cancelation.
* We start with a primal non-feasible solution, and send flow from positive to negative nodes.

Dual characterization of optimality

- **Relabeling:** On every node i, divide each f_{ij} by the same factor $\mu_i > 0$. Let $f_{ij}^{\mu} = f_{ij}/\mu_i$. *(Change from dollars to cents).*
- This gives an equivalent problem with $\gamma_{ij}^{\mu} = \gamma_{ij} \dfrac{\mu_i}{\mu_j}$
- The relabeling is conservative, if
  - in the residual graph, only lossy arcs: $\gamma_{ij}^{\mu} \leq 1$
  - for negative arcs, $\mu_i = 1/M_i$

Claim: f is optimal $\Leftrightarrow$ there exists no GAP $\Leftrightarrow$ there exists a conservative relabeling with all positive nodes i having $\mu_i = \infty$.

Concave generalized flow algorithm

* Use the symmetric generalized flow algorithm, with a linearization of gain functions in Δ-chunks, as by Minoux, Hochbaum&Shantikumar.

Concave symmetric formulation

$$\min \sum_{i \in V} M_i \kappa_i$$

$$\sum_{j:ji \in E} \Gamma_{ji}(f_{ji}) - \sum_{j:ij \in E} f_{ij} \geq b_i - \kappa_i \quad \forall i \in V$$

$$0 \leq f_{ij} \quad \forall ij \in E$$

$$0 \leq \kappa_i \quad \forall i \in V$$

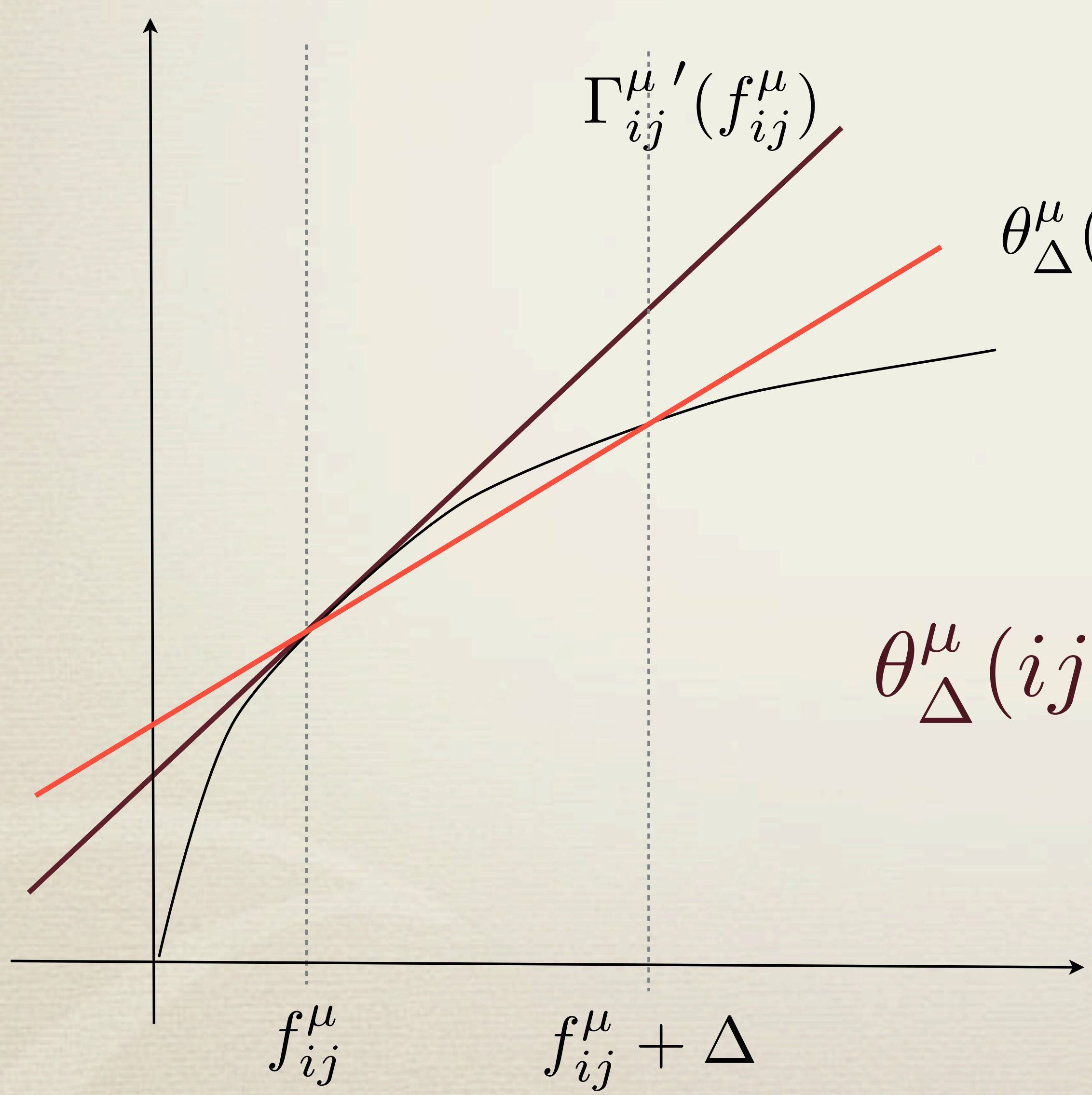
Dual characterization of optimality

- **Relabeling:** On every node i, divide each f_{ij} by the same factor $\mu_i > 0$. Let $f^{\mu}_{ij} = f_{ij}/\mu_i$ *(Change from dollars to cents)*.
- This gives an equivalent problem with $\Gamma^{\mu}_{ij}(\alpha) = \dfrac{\Gamma_{ij}(\mu_i \alpha)}{\mu_j}$
- The relabeling is **conservative**, if
  - in the residual graph, only lossy arcs: $\Gamma^{\mu\,\prime}_{ij}(f^{\mu}_{ij}) \leq 1$
  - for negative nodes, $\mu_i = 1/M_i$

Claim: f is optimal ⇔ there exists no GAP ⇔ there exists a conservative relabeling with all positive nodes i having $\mu_i = \infty$.

Δ-conservative labelings

* In the Δ-phase, we relax the condition on lossy arcs: instead of ${\Gamma^{\mu}_{ij}}'(f^{\mu}_{ij}) \leq 1$, we only require

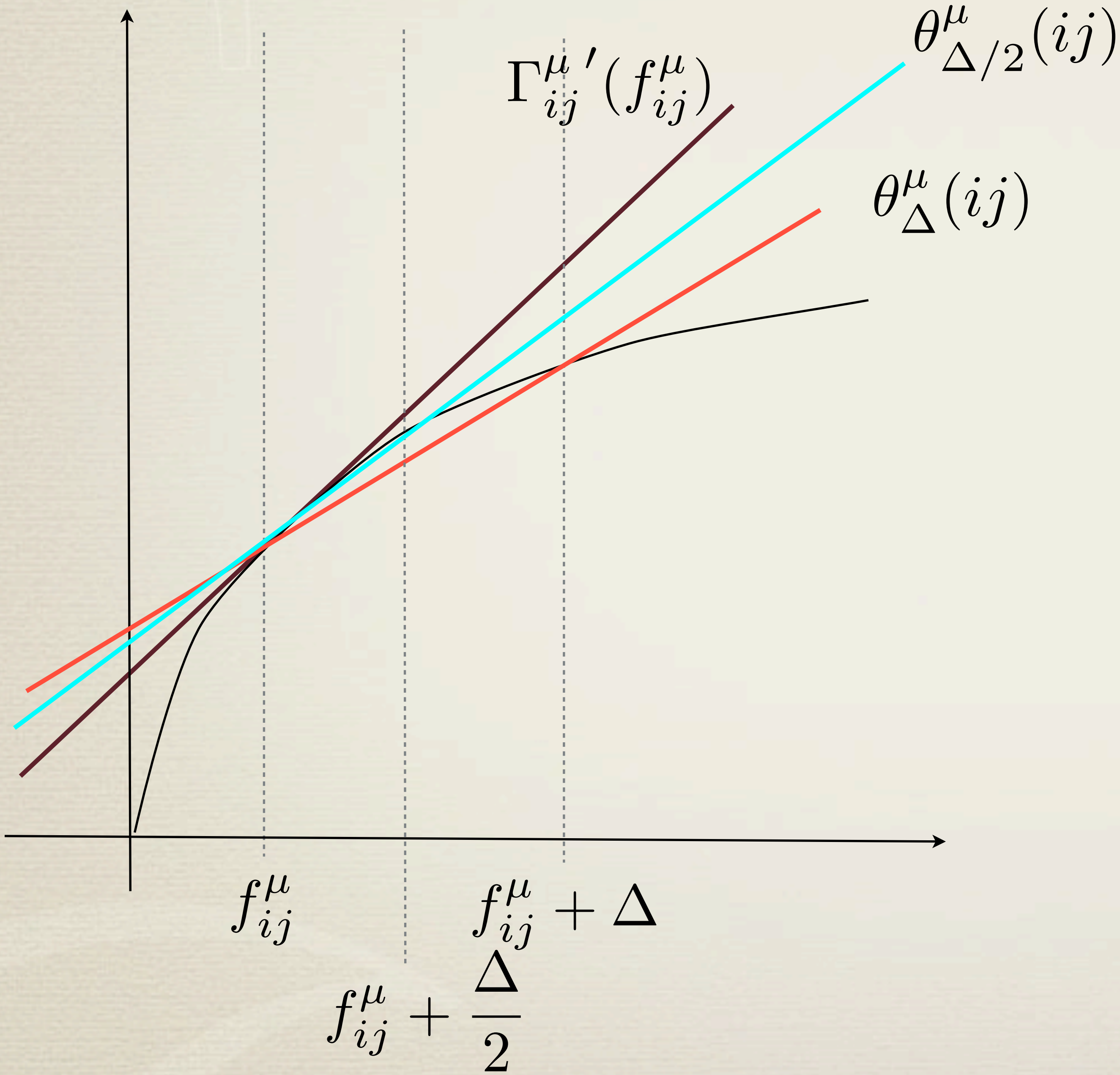


$$\theta^{\mu}_{\Delta}(ij) := \frac{\Gamma^{\mu}_{ij}(f^{\mu}_{ij} + \Delta) - \Gamma^{\mu}_{ij}(f^{\mu}_{ij})}{\Delta} \leq 1$$

Δ-conservative labelings

Lemma: Given a Δ-conservative labeling, if we send Δ units of flow on residual arcs with $\theta^{\mu}_{\Delta}(ij) = 1$, then the Δ-conservative property is maintained.

From Δ to $\Delta/2$ phase



Problem:
$\Delta/2$-conservativeness is stronger than Δ-conservativeness

Lemma: On every arc, we can change the relabeled flow by $\pm\Delta/2$ to obtain $\Delta/2$-conservativeness.

From Δ to $\Delta/2$ phase

Bigger problem: Adjustments might change positive nodes into negative, giving new negative nodes with $\mu_i > 1/M_i$

- ☐ If a node is negative at the end, we need $\mu_i = 1/M_i$
- ☐ During the algorithm, the labels μ_i may never decrease.

Solution:

Redefine negative!

From Δ to $\Delta/2$ phase

Originally: A node is negative, if its net flow is less than the demand b_i and serves as a sink in the algorithm.

Modification:

- A node is negative, if its net flow is less than the demand b_i plus a security reserve $d_i \Delta \mu_i$.
- The reserve compensates for all later adjustments and guarantees that any nonnegative node remains forever nonnegative.
- We also send flow to nodes where the demand is satisfied.

Further questions

* What are the limitations of combinatorial algorithms for convex optimization?
* When is there hope for strongly polynomial running time?
* V. ’12: strongly polynomial algorithm for a class of flows with separable convex objectives.
  * contains linear Fisher’s market based on a different convex formulation.

Further questions

* No strongly polynomial algorithm exists for linear generalized flows!
* Solving that could help develop strongly poly. alg. for certain concave gain functions.
* Strongly polynomial algorithm for **submodular** flows with separable convex objectives?

Thank you for your attention!